Design Methodology of Microstrip Lines Using Dimensional Analysis

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***Abstract*— Dimensional analysis has been used to develop a new methodology in microwave integrated circuit design using microstrip lines. The analysis reveals a complete set of dimensionless products for the microstrip line. The dependencies between these products are established using the Pi theorem. Using commercial simulation tools and numerical fitting techniques, design formulas can be established. These design formulas calculate the desired microwave parameters, such as characteristic impedance, effective dielectric constants, and their frequency dependencies.**

***Index Terms*—Dimensional analysis, methodology, microstrip, microwave design.**

I. Introduction

DIMENSIONAL analysis has been used by designers in numerous areas to simplify the analysis of an otherwise complicated problem. Previously, in solid mechanic and fluid dynamic systems, the principle of dimensional analysis is applied to relate the data obtained using a scaled-down model to the performance prediction of the actual-size system. In the microwave area, however, the dimensional analysis has not been used to take an advantage of its simplification characteristics. In this letter, we applied dimensional analysis as a new design methodology in the microwave design using a two-dimensional microstrip line as an example.

In order for the dimensional analysis to be valid, the equation describing a certain physical phenomenon needs to be of dimensional completeness and dimensional homogeneity [1], [2]. The equation is dimensionally complete if the equation expression remains unchanged when the size of the fundamental unit changes. The dimensional homogeneity is that any function of the dimensionless variable remains dimensionless. With these two properties, it was proven that the equation can be reduced to a relationship among a complete set of dimensionless products of the system variables. This is known as the Pi (π) theorem [1], [2], and the dimensionless product terms are known as π terms. Apparently, since all the scientific and engineering equations possess these two properties, the dimensional analysis is applicable.

Suppose there are n fundamental units of irreducible dimensionality, $u_1, u_2, \cdots$, and u_n. A physical system involves m system variables, $p_1, p_2, \cdots$, and p_m. Then, according to the principle of the absolute significance of relative magnitude it was proven that the unit of the system variable $[p_i]$ can be expressed by [1]

$$[p_i] = u_1^{b_{1i}} \quad u_2^{b_{2i}} \quad \cdots \quad u_n^{b_{ni}}$$

where $b_{1i}, b_{2i}, \cdots$, and b_{ni} are the exponents. A typical dimensionless product is given by

$$\pi = p_1^{a_1} \quad p_2^{a_2} \quad \cdots \quad p_m^{a_m}$$

where $a_1, a_2, \cdots$ and a_m are the exponents. The unit of π is given by

$$[\pi] = u_1^{\Sigma a_i b_{1i}} \quad u_2^{\Sigma a_i b_{2i}} \quad \cdots \quad u_n^{\Sigma a_i b_{ni}}$$

where the summation index in the exponent is running from 1 to m. Since π is dimensionless, a system of equations can be set up to solve for $a_1, a_2, \cdots$ and a_m as below, and the exponents can be obtained by solving the matrix equation $[B][A] = 0$ in the following:

$$\begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & b_{1m} \\ b_{21} & b_{22} & \cdot & \cdot & b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & b_{nm} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system of equations yields only a finite number of nontrivial independent solutions. The finite number of independent solutions, according to matrix algebra, is equal to the number of system variable minus the rank of the matrix B [1], [2]. Each independent solution results in a dimensionless π term. After all the π terms are determined, the dependency between the system parameters can be established according to Pi theorem.

To illustrate the application of dimensional analysis to solving the microwave problems, we will use a microstrip transmission line as an example. The following sections show our derivations to calculate the characteristic impedance of the microstrip line as a function of w/h and ε_r at low frequencies, and as a function of w/h and hf at high frequencies using GaAs substrate, where w is the width of the transmission line, h is dielectric height, ε_r is the relative dielectric constant, and f is the frequency.

II. Analysis of Microstrip Line

A. Microstrip Line System with Low-Frequency Approximation

Fig. 1 shows a cross section of a microstrip line. The substrate thickness is h_1, the thickness above the substrate

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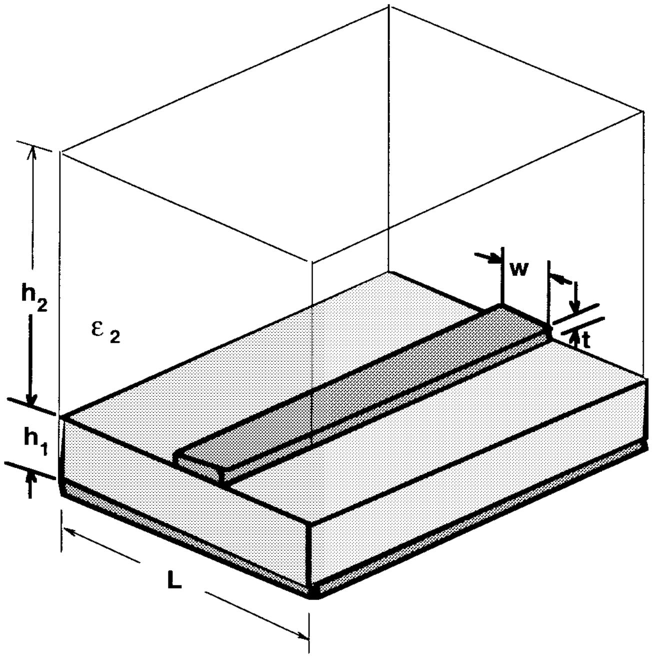

Fig. 1. A schematic diagram of the microstrip line. The thicknesses of the substrate and the ambient layer are h_1 and h_2, respectively. The dielectric constants for the substrate and the ambient layer is ε_1 and ε_2.

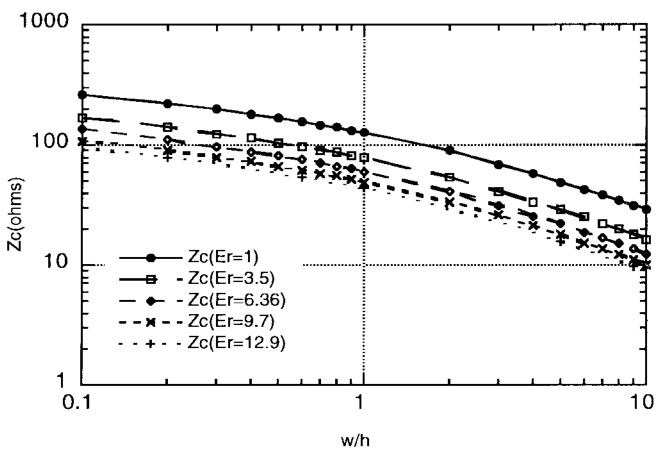

Fig. 2. The calculated results of the characteristic impedance as a function of w/h with relative dielectric constant as the parameter.

is h_2, the microstrip line width is w, and the lateral dimension is L. The dielectric constants in and above the substrate are ε_1 and ε_2, respectively. The bottom and the top horizontal surfaces of the domain are assuming electrical walls, and the two vertical surfaces are assuming magnetic walls. Under the approximation of the quasi-TEM, the characteristic impedance Z_c [3], [4] is defined to be the static voltage between the microstrip line and the ground plane divided by the static current in the microstrip line. These quantities are the static solutions to the Maxwell's equations, and they can be simulated and measured.

At low frequencies, wave propagation in the microstrip line can be approximated as quasi-TEM mode. Z_c approaches an asymptotic value as frequency f becomes small. In applying the dimensional analysis, the system variables and their corresponding units need to be determined. The system variables are $h_1, h_2, w, L, \varepsilon_1, \varepsilon_2$, and Z_c. The corresponding units of ε_1, for example, can be represented as farad/meter $= [M]^{-1}[L]^{-3}[T]^2[Q]^2$, where $[M], [L], [T]$, and $[Q]$ stand for the fundamental units of mass, length, time, and electrical charge, respectively. The exponents of each unit, $-1, -3, 2$, and 2 form the column vector of the following matrix $[B]$. A dimensional constant of c, the speed of light in a vacuum, is also called for in the analysis. Therefore, the matrix $[B]$ is derived in the following according to the Pi theorem:

$$\begin{array}{c|cccccccc} & h_1 & h_2 & w & L & \varepsilon_1 & \varepsilon_2 & c & Z_c \\ [M] & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ [L] & 1 & 1 & 1 & 1 & -3 & -3 & 1 & 2 \\ [T] & 0 & 0 & 0 & 0 & 2 & 2 & -1 & -1 \\ [Q] & 0 & 0 & 0 & 0 & 2 & 2 & 0 & -2. \end{array}$$

The rank of this matrix is three. The number of the π terms is, therefore, $8 - 3 = 5$. The five independent solutions of $[A]$ resulted in the following π terms:

$$\pi_1 = h_2/h_1, \quad \pi_2 = w/h_1, \quad \pi_3 = L/h_1$$
$$\pi_4 = \varepsilon_1/\varepsilon_2, \quad \pi_5 = c\varepsilon_2 Z_c.$$

According to the Pi theorem, the characteristic impedance Z_c becomes a dependent function ϕ of the π terms:

$$Z_c = \frac{1}{c\varepsilon_2}\phi\left(\frac{h_2}{h_1}, \frac{w}{h_1}, \frac{L}{h_1}, \frac{\varepsilon_1}{\varepsilon_2}\right).$$

B. Microstrip Line System with Frequency Dependence [3], [5]

As frequency increases, quasi-TEM approximation becomes less applicable, and the dispersive nature becomes increasingly enhanced. In the microwave circuit, the characteristic impedance Z_c of the dominant mode is no longer unique for the higher modes. To study the dispersion of Z_c at higher frequencies, the frequency f needs to be added into the set of the system variables. Applying the same procedures as before, one obtains the matrix $[B]$ given by

$$\begin{array}{c|ccccccccc} & h_1 & h_2 & w & L & \varepsilon_1 & \varepsilon_2 & c & f & Z_c \\ [M] & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ [L] & 1 & 1 & 1 & 1 & -3 & -3 & 1 & 0 & 2 \\ [T] & 0 & 0 & 0 & 0 & 2 & 2 & -1 & -1 & -1 \\ [Q] & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & -2. \end{array}$$

The rank of this matrix is again three. The number of the π terms is, therefore, 6 and they are

$$\pi_1 = h_2/h_1, \quad \pi_2 = w/h_1, \quad \pi_3 = L/h_1$$
$$\pi_4 = \varepsilon_1/\varepsilon_2, \quad \pi_5 = fh_1/c, \quad \pi_6 = c\varepsilon_2 Z_c.$$

The characteristic impedance Z_c from the Pi theorem becomes

$$Z_c = \frac{1}{c\varepsilon_2}\phi\left(\frac{h_2}{h_1}, \frac{w}{h_1}, \frac{L}{h_1}, \frac{\varepsilon_1}{\varepsilon_2}, \frac{fh_1}{c}\right).$$

A further simplification of the dependencies of Z_c on the design parameters will be followed.

III. Results and Discussion

In the practical consideration for the microstrip line design, one may further simplify the functional dependence for $Z_c, Z_c = \phi(\frac{w}{h}, \varepsilon_r, hf)$ in Section II-B. The calculated results of the characteristic impedance as a function of w/h with relative dielectric constant ε_r are shown in Fig. 2. The data of Fig. 2 was obtained from the simulated results using Microwave Harmonica developed by Super-Compact. Using least square fitting to the third order of $\log(w/h)$ and ε_r, Z_c at low hf can be expressed by

$$\begin{aligned} Z_c &= m_0(\varepsilon_r) + m_1(\varepsilon_r)\log(w/h) + m_2(\varepsilon_r)(\log(w/h))^2 \\ &\quad + m_3(\varepsilon_r)(\log(w/h))^3 \\ m_0(\varepsilon_r) &= 126.27 - 101.79\log(\varepsilon_r) + 24.25(\log(\varepsilon_r))^2 \\ m_1(\varepsilon_r) &= -129.67 + 91.32\log(\varepsilon_r) - 16.06(\log(\varepsilon_r))^2 \\ m_2(\varepsilon_r) &= 20.82 - 1.88\varepsilon_r + 0.13\varepsilon_r^2 - 0.0043\varepsilon_r^3 \\ m_3(\varepsilon_r) &= 16.48 - 3.55\varepsilon_r + 0.35\varepsilon_r^2 - 0.011\varepsilon_r^3. \end{aligned}$$

The correlation between m_0 and m_1 with ε_r is more than 99.9% while the correlation between m_2 and m_3 with ε_r are 97% and 94%, respectively. Therefore, the accuracy of this fitting is more than 99% for the lower range of the values of w/h and ε_r. However, for the values of w/h close to 10 and ε_r close to 12 the accuracy falls to 93%.

With this formula, the other important design parameter of effective dielectric constant ε_{eff} can be calculated by $\varepsilon_{\text{eff}} = (Z_c^0/Z_c)^2$, where Z_c^0 is the characteristic impedance when $\varepsilon_r = 1$.

At certain occasions, the inverse function of the previous formula for Z_c is desirable, that is, to find a w/h value for a desired characteristic impedance Z_c. The inverse formula is given in the following:

$$\begin{aligned} \log(w/h) &= m_0 + m_1(\varepsilon_r)\log(Z_c) + m_2(\varepsilon_r)\{\log(Z_c)\}^2 \\ &\quad + m_3(\varepsilon_r)\{\log(Z_c)\}^3 \\ m_0(\varepsilon_r) &= 11.567 - 2.1389\varepsilon_r + 0.222\,75\varepsilon_r^2 - 0.007\,843\,8\varepsilon_r^3 \\ m_1(\varepsilon_r) &= -16.04 + 2.6919\varepsilon_r - 0.276\,87\varepsilon_r^2 + 0.009\,712\,6\varepsilon_r^3 \\ m_2(\varepsilon_r) &= 8.492 - 1.0939\varepsilon_r + 0.111\,93\varepsilon_r^2 - 0.003\,925\,4\varepsilon_r^3 \\ m_3(\varepsilon_r) &= -1.6246 + 0.108\,97\varepsilon_r - 0.012\,291\varepsilon_r^2 \\ &\quad + 0.000\,442\,04\varepsilon_r^3. \end{aligned}$$

The accuracy of above relation for w/h is more than 90% for the range of $w/h = 0.1$ through 10.

The dependency of $h_1 f$ with $w/h = 0.25, 0.5, 1.0$, and 2.0 and $\varepsilon_r = 12.9$ is shown in Fig. 3. Z_c as a function of h_f and w/h with a GaAs substrate ($\varepsilon_r = 12.9$) is given in the following using the least square fitting technique

$$\begin{aligned} Z_c\ (w/h, hf) &= \gamma(hf) Z_c(w/h) \\ \gamma(hf) &= 1.000 - 0.001\,794\,4(hf) + 0.000\,571\,15(hf)^2 \end{aligned}$$

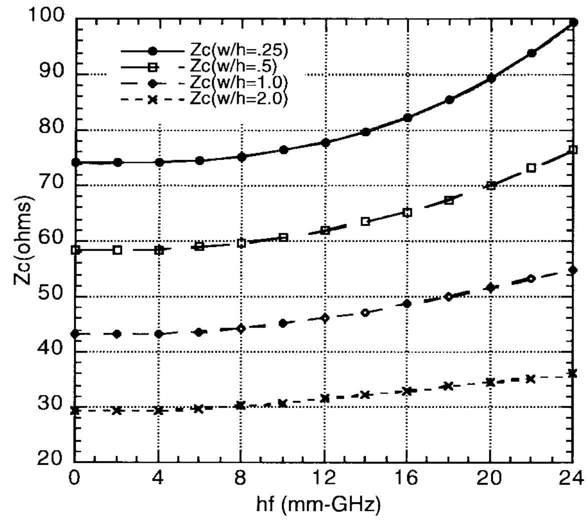

Fig. 3. The calculated results of the characteristic impedance as a function of hf with relative dielectric constant is 12.9 and w/h as the parameter.

where $Z_c(w/h)$ is a characteristic impedance at low frequencies and γ is a correction factor due to the dispersion. The accuracy of the fitting is more than 95%.

IV. Conclusion

Dimensional analysis was applied to develop a new design methodology in the microwave integrated circuit design area. This work is a first known application of dimensional analysis in the microwave integrated circuit design. This novel approach toward microwave design provides the insight for developing scalable microwave performance models. The results show a complete dependency of the dimensionless product terms for the microwave parameters of the characteristic impedance, effective dielectric constant, and the operating frequency, and leads to a scalable microstrip line model. This approach will benefit the microwave integrated circuit design area as a performance-bound design methodology.

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